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THE REFLECTION OF BEAMS OF INTERNAL GRAVITY WAVES AT A FLAT RIGID SURFACE†

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The linear problem of the reflection of a beam of monochromatic internal waves at a rigid inclined wall in an exponentially stratified liquid with viscosity and diffusion is considered. In such a medium there is, as well as the reflected wave, a boundary-layer flow at the plane with split spatial length-scales for the velocity and density variation. Viscosity and diffusion restrict the limiting value of the geometrical compression coefficient of the beam. The solution has no singularities at critical angles of reflection. Calculations for the reflected beam and boundary flow are performed for an incident beam radiated by a point mass source.

Modern models of internal waves have been developed for plane waves filling the whole of space [1, 2] and spatially localized wave beams which are the fundamental type of motion in viscous media [3, 4].

When wave beams are reflected in an ideal exponentially stratified liquid, geometrical compression causes the wave amplitudes to become singular at critical angles at which the reflected wave runs along the inclined base [5, 6]. Similar effects are also observed in the reflection of the wave field with a complex frequency composition. In an ideal liquid, without the use of the traditional Boussinesq approximation, as well as the geometrical compression there is also distortion of the spatial spectrum of the incident wave [7].

Because the mechanisms for the development of instability in stratified flows depend on the fine structure of the velocity and density profiles, there is great interest in investigating the problem of the reflection of beams of internal waves taking viscosity and diffusion into account. Because of the effects of *distsersion* (the difference between the coefficients of molecular diffusion and kinematic viscosity) the diffusivity and dynamic boundary layers formed at the reflecting surface can be characterized by different length-scales: this occurs in slow boundary flow, induced diffusion at an inclined wall [8], and in a laminar boundary layer in a continuously stratified liquid [9].

1. GENERAL RELATIONS FOR THE REFLECTED FIELD

Suppose that in an incompressible viscous linearly-stratified liquid an infinite rigid plane is situated at an angle φ to the horizontal, and that a beam of monochromatic internal waves is incident on it. A Cartesian system of coordinates (x, y, z) is connected to the liquid with the z axis directed opposite to the acceleration due to gravity. We consider a two-dimensional problem in which every quantity is independent of the y coordinate. The linearized hydrodynamic equations in the Boussinesq approximation and in the presence of saline diffusion can be written in the form [10]

$$\frac{\partial \upsilon_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \nu\Delta\upsilon_x, \quad \frac{\partial \upsilon_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \nu\Delta\upsilon_z - gs \tag{1.1}$$

$$\frac{\partial \upsilon_x}{\partial x} + \frac{\partial \upsilon_z}{\partial z} = 0, \quad \frac{\partial s}{\partial t} = D\Delta s + \frac{\upsilon_z}{\Lambda}, \quad \rho = \rho_0\left(1 - \frac{z}{\Lambda} + s\right)$$

Here (υ_x, υ_z), P and s are the velocity components, pressure and dimensionless salinity, which includes the saline compressibility coefficient, ν and D are the kinematic viscosity and saline diffusion coefficient, g is the acceleration due to gravity, Λ is the stratification length-scale, and ρ is the total density of the liquid.

Eliminating all the unknowns from (1.1) apart from υ_z and introducing the vertical liquid particle displacement h by the formula $\upsilon_z = \partial h/\partial t$, we obtain the equation

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$$\left[\left(\frac{\partial}{\partial t}-D\Delta\right)\left(\frac{\partial}{\partial t}-\nu\Delta\right)\Delta+N^2\frac{\partial^2}{\partial x^2}\right]h=0 \tag{1.2}$$

Suppose that a Cartesian system of coordinates (ξ, ζ) is attached to the plane with the ξ axis directed along the plane and the ζ axis normal to it, so that the change from the (x, z) system to the (ξ, ζ) system is performed by the formulae

$$x=\xi\cos\varphi+\zeta\sin\varphi, \quad z=\xi\sin\varphi-\zeta\cos\varphi \tag{1.3}$$

For a monochromatic wave $\partial/\partial t = -i\omega$ (which corresponds to a time dependence of the form exp $(-i\omega t)$ which is always omitted below), using the translational symmetry of the problem along the ξ axis, the general solution for the reflected field can be written in the form of a superposition of plane inhomogeneous waves of the form exp $[i(k_\xi\xi + k_\zeta\zeta 0)]$ with real k_ξ and complex k_ζ. As a result we obtain a dispersion equation relating k_ξ and k_ζ

$$\left[i\omega-D\left(k_\xi^2+k_\zeta^2\right)\right]\left[i\omega-\nu\left(k_\xi^2+k_\zeta^2\right)\right]\left(k_\xi^2+k_\zeta^2\right)+N^2(k_\xi\cos\varphi+k_\zeta\sin\varphi)^2=0 \tag{1.4}$$

For every real $k_\xi \equiv k$ Eq. (1.4) has six solutions $k_\zeta = k_j(k)$ with numbering chosen so that the inequalities $\operatorname{Im} k_3 > \operatorname{Im} k_2 > \operatorname{Im} k_1 > 0$ and $\operatorname{Im} k_6 < \operatorname{Im} k_5 < \operatorname{Im} k_4 < 0$ are satisfied. Direct calculations show that when $k \to 0$ the roots of Eq. (1.4) form two triplets with imaginary parts of opposite sign. This assertion also holds for all values of k. Indeed, if any of the roots k_j changes the sign of its imaginary part at some finite $k = k_0$, then it follows from (1.4) that $k_j^2 + k_0^2 = 0$, which is a contradiction.

In an ideal (inviscid) liquid without saline diffusion the boundary conditions at the rigid plane consist of the vanishing of the normal component of the total velocity, and the differential equation describing that the wave motion is of second order. This enables us to represent the reflected wave in the form

$$h=\int B(k)e^{i(k\xi+ik_1\zeta)}dk$$

with a single amplitude function $B(k)$ which should be determined from the boundary conditions [7]. Here and below, unless otherwise stated, integration is performed from $-\infty$ to $+\infty$.

In the two-dimensional formulation of the problem for a viscous liquid with saline diffusion three scalar boundary conditions appear. They can be satisfied if the reflected field has three independent amplitude functions

$$h=\sum\int B_j(k)e^{i(k\xi+ik_j\zeta)}dk \tag{1.5}$$

Summation is always performed from $j = 1$ to $j = 3$, and the numbering of the roots k_j of Eq. (1.4) has been given above. The amplitudes $B_1(k), B_2(k)$ and $B_3(k)$ should be determined from the condition that the normal and tangential components of the total velocity and the normal component of the salinity gradient all vanish.

Using the relation $\upsilon_2 = -i\omega h$ and the third equation from (1.1) we find the velocity component υ_x, and then we determine the salinity s from the fourth equation in (1.1). The velocity components (υ_x, υ_z) are coupled to the components $(\upsilon_\xi, \upsilon_\zeta)$ by the same relation (1.3) as for the coordinates, and so we finally obtain

$$\upsilon_\xi=-i\omega\sum\int\frac{k_jB_j(k)}{\beta_j}e^{i(k\xi+k_j\zeta)}dk, \quad \upsilon_\zeta=i\omega\sum\int\frac{kB_j(k)}{\beta_j}e^{i(k\xi+k_j\zeta)}dk \tag{1.6}$$

$$s=\frac{i\omega}{\Lambda}\sum\int\frac{B_j(k)}{\gamma_j}e^{i(k\xi+k_j\zeta)}dk, \quad \beta_j=k\cos\varphi+k_j\sin\varphi, \quad \gamma_j=i\omega-D(k^2+k_j^2)$$

If the distribution of velocity and salinity in the beam incident on the plane is given by $\upsilon_{0\xi}(\xi, \zeta)$ and $\upsilon_{0\zeta}(\xi, \zeta)$, and $s_0(\xi, \zeta)$ then the boundary conditions given above lead to a system of equations for finding the amplitudes $B_j(k)$. Performing an inverse Fourier transform we obtain a linear algebraic system for the $B_j(k)$ which, on solving, we obtain

$$B_i = \beta_i \gamma_i \Sigma \frac{A_{ij}(k)}{A(k)} F_j(k) \tag{1.7}$$

Here

$$F_1 = \frac{1}{2\pi i\omega} \int \upsilon_{0\xi}(\xi,0) e^{-ik\xi} d\xi, \quad F_2 = -\frac{1}{2\pi i\omega} \int \upsilon_{0\zeta}(\xi,0) e^{-ik\xi} d\xi$$

$$F_3 = \frac{\Lambda}{2\pi\omega} \int \frac{\partial s_0(\xi,0)}{\partial\zeta} e^{-ik\xi} d\xi$$

$$A_{i1} = -k(k_n - k_m)\{[i\omega - D(k^2 - k_n k_m)]k\cos\varphi + (i\omega - Dk^2)(k_n + k_m)\sin\varphi\}$$

$$A_{i2} = k_n k_m (k_n - k_m)\{[i\omega - D(k^2 - k_n k_m)]\sin\varphi + Dk(k_n + k_m)\cos\varphi\} \tag{1.8}$$

$$A_{i3} = k(k_n - k_m)[i\omega - D(k^2 + k_n^2)]\ [i\omega - D(k^2 + k_n^2)]$$

$$A = -k(k_1 - k_2)(k_2 - k_3)(k_3 - k_1)\{(i\omega - Dk^2)[i\omega - D(k^2 - k_1k_2 - k_2k_3 - k_3k_1)]\sin\varphi +$$

$$+Dk[(i\omega - Dk^2)(k_1 + k_2 + k_3) + Dk_1k_2k_3]\cos\varphi\}$$

and permutations of (i, n, m) are equivalent to the cyclic permutation (1, 2, 3).

Relations (1.5) and (1.7), (1.8) completely solve the problem of the reflection of internal wave beams at a rigid plane is a viscous liquid with saline diffusion.

2. ANALYSIS OF THE STRUCTURE OF THE SOLUTION

To explain the physical meaning of the terms occurring in solution (1.5), we must solve the dispersion equation (1.4) and find the complex wave numbers $k_1(k)$, $k_2(k)$ and $k_3(k)$. In the general case these roots cannot be found analytically because of the high (sixth) degree of Eq. (1.4), but for small values of the viscosity and diffusion coefficient one can use standard methods [11] to obtain the first terms of their asymptotic expansions

$$k_1 = -k\,\mathrm{ctg}(\varphi \pm \theta) \pm \frac{i(\nu + D)k^3}{2N\cos\theta\sin^4(\varphi \pm \theta)} \tag{2.1}$$

$$k_2 = (\pm 1 + i)\left[\frac{N^2|\sin^2\theta - \sin^2\varphi|}{\Omega}\right]^{1/2}, \quad k_3 = \frac{1+i}{2}\left[\frac{\Omega}{\nu D}\right]^{1/2}$$

$$\Omega = \omega(\nu + D) + [\omega^2(\nu - D)^2 + 4\nu DN^2\sin^2\varphi]^{1/2} \tag{2.2}$$

The upper sign in (2.1) is used when $k > 0$ and the lower sign when $k < 0$, the plus or minus sign in the second of formulae (2.1) is the same as the sign of the difference $\sin^2\theta - \sin^2\varphi$, and $\theta = \arcsin(\omega/N)$ is the angle of the internal wave beam to the horizontal. For propagating waves $0 < \omega/N < 1$ and, consequently, $0 < \theta < \pi/2$.

The first term in the summation (1.5) can be represented using (2.1) in the form

$$h_1 = \int_{-\infty}^{+\infty} B_1(k)\exp\{ik[\xi + \zeta\,\mathrm{ctg}(\varphi \pm \theta)]\}\exp\left[-\frac{(\nu + D)|k|^3\zeta}{2N\cos\theta\sin^4(\varphi \pm \theta)}\right]dk \tag{2.3}$$

Integrals of this form describe [1, 3] a diverging beam of interval waves whose transverse shape is governed by the spectral function $B_1(k)$. In this problem the degree of divergence of such a beam is governed by the sum of kinetic coefficients $\nu + D$, unlike the purely viscous case where $D = 0$, so that when there is saline diffusion the beam spreads out more rapidly.

The second and third terms in (1.5), taking relations (2.2) into account, acquire the form

$$h_j = \exp\left(\pm\frac{i\zeta}{\lambda_\pm}\right)\exp\left(-\frac{\zeta}{\lambda_\pm}\right)\int B_j(k)e^{ik\xi}dk,\ j=2,3$$

It is clear that the reflected field is a product of oscillating and exponentially damped functions of ζ with a function of ξ determined by the shape of the beam incident on the plane. This enables us to interpret the second and third terms in the summation [6] as boundary layers described by spatial scales $\lambda_+ = 1/\mathrm{Im}\,k_2$ and $\lambda_2 = 1/\mathrm{Im}\,k_3$. Introducing the dissipative spatial scales [12]

$$l_\nu = (\nu/N)^{1/2},\quad l_D = (D/N)^{1/2} \tag{2.4}$$

we can write

$$\lambda_+ = \lambda|\sin^2\varphi - \sin^2\theta|^{-1/2},\quad \lambda_- = 2l_\nu l_D/\lambda$$

$$\lambda \equiv \left\{\left(l_\nu^2 + l_D^2\right)\sin\theta + \left[\left(l_\nu^2 - l_D^2\right)^2\sin^2\theta + 4l_\nu^2 l_D^2\sin^2\varphi\right]^{1/2}\right\}^{1/2} \tag{2.5}$$

Comparison of λ_+ and λ_- shows that the inequality $\lambda_+ \geqslant \lambda_-$ is always satisfied, with equality only achieved when $\nu = D$ and $\sin\varphi = 0$ simultaneously, i.e. with a horizontal bottom. In the other cases the scales differ, and the difference can be very great. In the frequently encountered case when one of the kinetic coefficients substantially exceeds the other (for example, $\nu \gg D$), we can obtain much simpler expressions for the scales λ_+ and λ_- from (2.5)

$$\lambda_+ = l_\nu(2\sin\theta/|\sin^2\varphi - \sin^2\theta|)^{1/2},\quad \lambda_- = l_D(2/\sin\theta)^{1/2} \tag{2.6}$$

i.e. one of the boundary layers is exclusively associated with the presence of viscosity, and the other with the presence of saline diffusion. In the opposite case $\nu \ll D$ one must exchange the positions of l_ν and l_D in formulae (2.6).

It is clear from (2.1) and (2.2) that both the wave field and one of the boundary layers have a singularity when $\sin\varphi = \pm\sin\theta$. This singularity is of a fundamental nature when there is no viscosity and saline diffusion (where there are in fact no boundary layers) and expresses itself in an infinite transverse contraction of the reflected wave beam and an infinite increase in its amplitude when the plane coincides with the direction of propagation of the beam.

Note, however, that in Eqs (1.1) the kinetic coefficients are in front of the highest order spatial derivatives. Hence, as the field gradients increase in the reflected beam the effects of viscosity and diffusion come into the foreground. We can therefore expect that the consistent inclusion of these effects will remove these singularities. Indeed, putting $\sin\varphi = \mu\sin\theta$, $\mu = \pm 1$ in Eq. (1.4), we obtain its solution

$$\begin{gathered}
k_1 = -\mu k\,\mathrm{ctg}\,2\theta + \frac{i\mu(\nu+D)k^3\sin\theta}{N\sin^5 2\theta},\quad \mu k > 0 \\
k_1 = \frac{\sqrt{3}+i}{2}(-K)^{1/3},\quad \mu k < 0;\quad K = \frac{2\mu k N\cos\theta}{\nu + D} \\
k_2 = iK^{1/3},\quad \mu k > 0;\quad k_2 = \frac{-\sqrt{3}+i}{2}(-K)^{1/3},\quad \mu k < 0 \\
k_3 = (1+i)\left[\frac{(\nu+D)N\sin\theta}{2\nu D}\right]^{1/2}
\end{gathered} \tag{2.7}$$

It is clear that although the functions $k_j(k)$ given by formulae (2.7) are more complex than (2.1) and (2.2), there is no singularity when $\sin\varphi = \pm\sin\theta$. Here the spatial structure of one of the boundary layers has a complicated nature (not being a simple exponential), depending on how the beam is incident on the plane. At the same time the structure of the other boundary layer continues to be described by an exponential function for all position angles of the reflecting plane.

3. REFLECTION OF A WAVE BEAM RADIATED BY A POINT MASS SOURCE

As an example, consider a wave beam radiated by a point mass source located at the point $x = L \cos\theta, z = -L \sin\theta$, where L is the distance from the source to the plane along the direction of propagation of the beam. To fix our ideas, we will consider a beam propagating to the left and upwards. Here the angle φ lies in the range $-\theta < \varphi < \pi - \theta$, otherwise the beam will simply not be incident on the plane. In a rectangular Cartesian system of coordinates (p, q) with the q axis directed along the beam, and related to the (x, z) system by the formulae

$$x = -(q-L)\cos\theta - p\sin\theta, \quad z = (q-L)\sin\theta - p\cos\theta \tag{3.1}$$

the vertical displacement of particles in such a beam can be expressed by the integral [3]

$$h_0(p,q) = \int_0^\infty \exp\left[ikp - \frac{(\nu+D)k^3 q}{2N\cos\theta}\right]dk \tag{3.2}$$

Changing to the (ξ, ζ) system of coordinates using (1.3) and (3.1), and then integrating in the complex τ plane, from (3.2) we obtain

$$h_0(\xi,\zeta) = -\frac{1}{\sin(\varphi+\theta)} \int_C a(\tau)e^{i\tau\xi}e^{i\tau_1(\tau)\zeta}d\tau \quad a(\tau) = \exp[A(\tau)L]$$

$$\tau_1(\tau) = \frac{-\tau\cos(\varphi+\theta) + A(\tau)}{\sin(\varphi+\theta)}, \quad A(\tau) = \frac{(\nu+D)\tau^3}{2N\cos\theta\sin^3(\varphi+\theta)} \tag{3.3}$$

where the contour of integration C is given by the parametric equation

$$\tau(k) = -k\sin(\varphi+\theta) - \frac{i(\nu+D)k^3\cos(\varphi+\theta)}{2N\cos\theta}, \quad k \in [0, +\infty] \tag{3.4}$$

Deforming the contour C into the contour $\operatorname{Im}\tau = 0$ we obtain an expression for the vertical displacement of a particle in the incident beam in (ξ, ζ) coordinates

$$h_0(\xi,\zeta) = \frac{I(\xi,\zeta)}{\sin(\varphi+\theta)}, \quad I(\xi,\zeta) = \int_{-\infty}^0 a(\tau)e^{i\tau\xi}e^{i\tau_1(\tau)\zeta}d\tau \tag{3.5}$$

Using the relations $\upsilon_z = -i\omega h_0$, $\partial\upsilon_\xi/\partial\xi + \partial\upsilon_\zeta/\partial\zeta = 0$ and the relation between the components (υ_x, υ_z) and $(\upsilon_\xi, \upsilon_\zeta)$, which is identical with relation (1.3) between the coordinates (x, z) and (ξ, ζ), we find the velocity components

$$\upsilon_{0\xi} = \frac{i\omega\,\mathrm{ctg}(\varphi+\theta)}{\sin\theta}I(\xi,\zeta), \quad \upsilon_{0\zeta} = \frac{i\omega}{\sin\theta}I(\xi,\zeta) \tag{3.6}$$

From the third equation of system (1.1) we find the salinity in the incident beam

$$s_0 = \frac{i\omega}{\Lambda\sin(\varphi+\theta)} \int_{-\infty}^0 \frac{a(\tau)}{i\omega - D(\tau^2+\tau_1^2)} e^{i\tau\xi}e^{i\tau_1(\tau)\zeta}d\tau \tag{3.7}$$

Substitution of (3.6) and (3.7) into (1.7) and (1.8), taking into account the approximate solutions (2.1) and (2.2), enables us to find the spectral functions of the reflected wave field $B_1(k)$ and the boundary layers $B_2(k)$ and $B_3(k)$

$$B_1(k)=\frac{\vartheta(-k)}{\sin(\varphi-\theta)}\exp\left[-\frac{(\nu+D)k^3L}{2N\cos\theta}\right]$$

$$B_2(k)=b(k_2,k_3)B_1(k),\quad B_3(k)=b(k_3,k_2)B_1(k) \tag{3.8}$$

$$b(x,y)\equiv\frac{2\sin\varphi\cos\theta}{\sin(\varphi+\theta)}\frac{y(i\omega-Dx^2)}{(x-y)(i\omega+Dxy)}$$

where ϑ is the Heaviside unit function.

Substituting (3.8) into (1.5) we obtain an expression for the reflected wave field

$$h_\omega=\frac{1}{\sin(\varphi-\theta)}\int_{-\infty}^{0}\exp\left[\frac{(\nu+D)k^3L}{2N\cos\theta\sin^3(\varphi+\theta)}\right]e^{i(k\xi+k_1(k)\zeta)}dk \tag{3.9}$$

When $-\theta<\varphi<\theta$ the reflected beam will propagate to the left and downwards, and when $\theta<\varphi<\pi-\theta$, it will propagate to the right and upwards.

We introduce a rectangular system of coordinates (p,q) with q axis directed along the reflected beam and related to the (x,z) system by the relations

$$x=\mp(p\sin\theta+q\cos\theta),\quad z=\pm(p\cos\theta-q\sin\theta) \tag{3.10}$$

where the upper signs are taken for the first case and the lower signs for the second. We then obtain

$$h_w=\mp\int_0^\infty\exp\left[ikp-\frac{(\nu+D)k^3(q+L')}{2N\cos\theta}\right]dk,\quad L'=L\left|\frac{\sin(\varphi-\theta)}{\sin(\varphi+\theta)}\right|^3 \tag{3.11}$$

Thus the beam reflected from the plane is equivalent to a beam produced by the original source situated behind the plane at a distance L' along the direction of propagation of the reflected beam. Because the phases of beams propagating upwards radiated by a point mass source are opposite to the phases of beams propagating downwards, then when $-\theta<\varphi<\theta$ the phase of the reflected beam is opposite to the phase of the incident beam, whereas when $\theta<\varphi<\pi-\theta$ the incident and reflected beams are in phase. At the critical angle $\varphi=\theta$, when the reflected beam propagates along the reflecting plane, its phase structure is changed.

The above analysis shows that viscosity and diffusion affect the nature of the reflection of internal wave beams by being associated with the formation of a split boundary flow on the reflecting surface and by limiting the coefficient of geometrical contraction at the critical angles.

In the boundary flow, which is formed at the reflecting surface, the scales of spatial variation of the velocity and salinity (density) are always different if the coefficient of molecular momentum transfer and mass transfer differ (i.e. a medium with distsersion). The ratio of the scales also depends on the geometry of the problem and is given by formulae (2.4) and (2.5). Even in the degenerate case, when the coefficients of kinematic viscosity and diffusion are equal, the typical thicknesses of the density and dynamic boundary layers differ from one another except for the case of a horizontal reflecting surface.

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